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3.2 a) Let $\theta = \#$ of defectives. Set $f(\theta, K) = P(\text{accepting the lot} \mid \theta)$

Then $f(\theta, K) = \binom{N-\theta}{K}\binom{\theta}{0} \Big/ \binom{N}{K}$.

(3) Note $\dfrac{f(\theta+1, K)}{f(\theta, K)} = \dfrac{N-\theta-K}{N-\theta} < 1$; hence $f \downarrow$ in θ for fixed K and $\downarrow$ in K.

If $\theta \geq 6$, $f(\theta, K) \leq f(6, K)$ for $\theta \geq 6$.
We find K such that $f(6, K) \leq 0.1$ By trial and error the smallest K is $K = 32$

(3) b) Write $g(\theta, K) = \left[\binom{100-\theta}{K} + \binom{100-\theta}{K-1}\binom{\theta}{1}\right] \Big/ \left[\binom{100}{K}\right]$

As before we can show $g \downarrow \theta$

The smallest value of K for which $g(\theta, K) < 0.1$ is $K = 51$.

3.21 The mgf is $E e^{tX} = \displaystyle\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{e^{tx}}{1+x^2}\, dx$

3) On $(0, \infty)$, $e^{tx} > x$ and hence $\displaystyle\int_0^\infty \frac{e^{tx}}{1+x^2}\, dx > \int_0^\infty \frac{x}{1+x^2}\, dx = \infty$

Hence the mgf does not exist as expected since the moments of the Cauchy do not exist.

3.32 $f(x|\eta) = h(x)\, c(\eta) \exp\left(\sum_1^k \eta_i t_i(x)\right)$. Since f is a density,

$$1 = \int c(\eta)\, h(x) \exp\left(\sum_1^k \eta_i t_i(x)\right) dx$$

Differentiate both sides with respect to η_j (this can be done for the exponential family) twice.

① $$0 = \frac{\partial}{\partial \eta_j} \int c(\eta)\, h(x) \exp\left(\sum \eta_i t_i(x)\right) dx$$

(3) $$= \int \left[h(x)\right]\left[\frac{\partial c(\eta)}{\partial \eta_j} + t_j(x)\right] \exp\left(\sum \eta_i t_i(x)\right) dx$$

$$\Rightarrow \quad 0 = \left[\int h(x)\, c(\eta) \exp\left(\sum \eta_i t_i(x)\right) dx\right] \frac{1}{c(\eta)} \frac{\partial c(\eta)}{\partial \eta_j} + \int t_j(x) f(x|\eta)\, dx$$

$$\Rightarrow \quad E\, t_j(X) = -\frac{\partial}{\partial \eta_j} \log c(\eta)$$

② $$0 = \int \left[h(x)\right] \left[\frac{\partial^2 c(\eta)}{\partial \eta_j^2}\right] \exp\left(\sum \eta_i t_i(x)\right) dx + \int \left[h(x)\right] \left[\frac{\partial c(\eta)}{\partial \eta_j} + t_j(x)\right] t_j(x) \exp\left(\sum \eta_i t_i(x)\right) dx$$

Noting that $\frac{\partial^2 \ln c}{\partial \eta_j}(\eta) = -\frac{1}{c^2} \frac{\partial c(\eta)}{\partial \eta_j} + \frac{1}{c} \frac{\partial^2 c(\eta)}{\partial \eta_j^2}$ and

$\operatorname{Var} t_j(X) = E\, t_j^2(X) - \left[E\, t_j(X)\right]^2$ yields the result

b) Let $\sigma_1 > \sigma_2$, $X_1 \sim f(x/\sigma_1)$, $X_2 \sim f(x/\sigma_2)$. Let $F(x)$ be the cdf corresponding to $f(x)$, and $Z \sim f(x)$. Then

$$F(x/\sigma_1) = P(X_1 \le x) = P(\sigma_1 Z \le x) = F(x/\sigma_1) \le F(x/\sigma_2) = P(Z \le x/\sigma_2)$$

(3)

$$= P(X_2 \le x) \qquad = F(x|\sigma_2)$$

To get strict inequality for some x, let $(a, b]$ be an interval such that $a > 0$, $b/a = \sigma_1/\sigma_2$) and $P(a < Z \le b) = F(b) - F(a) > 0$. Let $x = a\sigma_1$. Then

$$F(x|\sigma_1) = F\left(\frac{x}{\sigma_1}\right) = F\left(\frac{a\sigma_1}{\sigma_1}\right) = F(a) < F(b)$$

$$= F\left(\frac{a\sigma_1}{\sigma_2}\right) = F\left(\frac{x}{\sigma_2}\right) = F(x|\sigma_2)$$

5.3 $Y_i \sim$ Bernoulli with $p_i = P(X_i \ge \mu) = 1 - F(\mu)$. Since the Y_i are iid,

(2) $\sum_1^n Y_i \sim \text{binomial}(n, p)$ where $p = 1 - F(\mu)$

5.6 a) For $Z = X - Y$ set $W = X$. Then $J = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = 1$

(3) and $f_{Z,W}(z,w) = f_X(w) \cdot f_Y(w - z) \cdot 1$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(w-z)\, dw$$

b) For $Z = XY$, set $W = X$. Similarly we find

$$f_{Z,W}(z,w) = f_X(w) f_Y(z/w) \left| \frac{1}{w} \right| \quad \text{and}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w) \, dw$$

c) For $Z = X/Y$, set $W = X$. Similarly we find

$$f_{Z,W}(z,w) = f_X(w) f_Y(w/z) \left| w/z^2 \right|$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(w/z) \left| \frac{w}{z^2} \right| dw$$